Existence of solutions to boundary value problem of fourth-order with functional boundary conditions at resonance

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Abstract: We study the existence of solutions for a fourth-order functional boundary value problem at resonance $\begin{cases} u^{(4)}(t) = f(t,u(t),u'(t),u''(t),u'''(t),),t \in (0,1) \\ \varphi_1(u) = \varphi_2(u) = \varphi_3(u) = \varphi_4(u) = 0 \end{cases}$ where $\varphi_i : C^3[0,1] \to R, i = 1,2,3$. By using the coincidence degree theory due to Mawhin and constructing suitable operators.

1. Introduction and introduction

A boundary value problem is said to be at resonance if the corresponding homogeneous boundary value problem has a non-trivial solution. Boundary value problems at resonance have been studied bymany authors. .We refer the readers to [1-9] and the references cited therein. In [10], the authors discussed the second-order differential equation $x''(t) = f(t, x(t), x'(t)), t \in (0,1)$ with functional boundary conditions $\Gamma_1(x) = 0, \Gamma_2(x) = 0$, where Γ_1, Γ_2 are linear functional on $C^1[0,1]$ satisfying the general resonance condition $\Gamma_1(x)\Gamma_2(1) = \Gamma_1(1)\Gamma_2(x)$.

In [11] proved the existence of solutions for third-order functional boundary value problems (FBVPs) at resonance

$$\begin{cases} x'''(t) = f(t, x(t), x'(t), x''(t)), 0 < t < 1 \\ \varphi_1(x) = \varphi_2(x) = \varphi_3(x) = 0, \end{cases}$$

where $\varphi_i : C^3[0,1] \to R, i = 1,2,3,4$ are bounded linear functionals. In this paper, the existence of solutions to the following boundary value problems is studied by using the coincidence degree extension theorem

$$\begin{cases} u^{(4)}(t) = f(t,u(t),u'(t),u''(t),u'''(t),),t \in (0,1) \\ \varphi_1(u) = \varphi_2(u) = \varphi_3(u) = \varphi_4(u) = 0 \end{cases} \quad (1)$$

where $\varphi_i : C^3[0,1] \to R, i = 1,2,3$, $\varphi_i(t^j) = 0, i = 1,2,3,4, j \in \{1,2,3,4\}$.

2. Preliminaries

For convenience, we denote

$$\Delta = \begin{vmatrix} \varphi_1(t^3) & \varphi_1(t^2) & \varphi_1(t) & \varphi_1(1) \\ \varphi_2(t^3) & \varphi_2(t^2) & \varphi_2(t) & \varphi_2(1) \\ \varphi_3(t^3) & \varphi_3(t^2) & \varphi_3(t) & \varphi_3(1) \\ \varphi_4(t^3) & \varphi_4(t^2) & \varphi_4(t) & \varphi_4(1) \end{vmatrix},$$

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$$\Delta_1(v)=\begin{vmatrix}\varphi_1(\int_0^t(t-s)^3v(s)ds) & \varphi_1(t^2) & \varphi_1(t) & \varphi_1(1)\\ \varphi_2(\int_0^t(t-s)^3v(s)ds) & \varphi_2(t^2) & \varphi_2(t) & \varphi_2(1)\\ \varphi_3(\int_0^t(t-s)^3v(s)ds) & \varphi_3(t^2) & \varphi_3(t) & \varphi_3(1)\\ \varphi_4(\int_0^t(t-s)^3v(s)ds) & \varphi_4(t^2) & \varphi_4(t) & \varphi_4(1)\end{vmatrix},$$

$$\Delta_2(v)=\begin{vmatrix}\varphi_1(t^3) & \varphi_1(\int_0^t(t-s)^3v(s)ds) & \varphi_1(t) & \varphi_1(1)\\ \varphi_2(t^3) & \varphi_2(\int_0^t(t-s)^3v(s)ds) & \varphi_2(t) & \varphi_2(1)\\ \varphi_3(t^3) & \varphi_3(\int_0^t(t-s)^3v(s)ds) & \varphi_3(t) & \varphi_3(1)\\ \varphi_4(t^3) & \varphi_4(\int_0^t(t-s)^3v(s)ds) & \varphi_4(t) & \varphi_4(1)\end{vmatrix},$$

$$\Delta_3(v)=\begin{vmatrix}\varphi_1(t^3) & \varphi_1(t^2) & \varphi_1(\int_0^t(t-s)^3v(s)ds) & \varphi_1(1)\\ \varphi_2(t^3) & \varphi_2(t^2) & \varphi_2(\int_0^t(t-s)^3v(s)ds) & \varphi_2(1)\\ \varphi_3(t^3) & \varphi_3(t^2) & \varphi_3(\int_0^t(t-s)^3v(s)ds) & \varphi_3(1)\\ \varphi_4(t^3) & \varphi_4(t^2) & \varphi_4(\int_0^t(t-s)^3v(s)ds) & \varphi_4(1)\end{vmatrix},$$

$$\Delta_4(v)=\begin{vmatrix}\varphi_1(t^3) & \varphi_1(t^2) & \varphi_1(t) & \varphi_1(\int_0^t(t-s)^3v(s)ds1)\\ \varphi_2(t^3) & \varphi_2(t^2) & \varphi_2(t) & \varphi_2(\int_0^t(t-s)^3v(s)ds)\\ \varphi_3(t^3) & \varphi_3(t^2) & \varphi_3(t) & \varphi_3(\int_0^t(t-s)^3v(s)ds)\\ \varphi_4(t^3) & \varphi_4(t^2) & \varphi_4(t) & \varphi_4(\int_0^t(t-s)^3v(s)ds)\end{vmatrix},$$

From the last three determinants we can define and derive the following three relations:

$$\Delta_1(Lu)=\begin{vmatrix}\varphi_1(-u'''(0)t^3-2u''(0)t^2-2u'(0)t-2u(0)) & \varphi_1(t^2) & \varphi_1(t) & \varphi_1(1)\\ \varphi_2(-u'''(0)t^3-2u''(0)t^2-2u'(0)t-2u(0)) & \varphi_2(t^2) & \varphi_2(t) & \varphi_2(1)\\ \varphi_3(-u'''(0)t^3-2u''(0)t^2-2u'(0)t-2u(0)) & \varphi_3(t^2) & \varphi_3(t) & \varphi_3(1)\\ \varphi_4(-u'''(0)t^3-2u''(0)t^2-2u'(0)t-2u(0)) & \varphi_4(t^2) & \varphi_4(t) & \varphi_4(1)\end{vmatrix}=-u'''(0)\Delta \qquad (2)$$

$$\Delta_2(Lu)=\begin{vmatrix}\varphi_1(t^3) & \varphi_1(-u'''(0)t^3-2u''(0)t^2-2u'(0)t-2u(0)) & \varphi_1(t) & \varphi_1(1)\\ \varphi_2(t^3) & \varphi_2(-u'''(0)t^3-2u''(0)t^2-2u'(0)t-2u(0)) & \varphi_2(t) & \varphi_2(1)\\ \varphi_3(t^3) & \varphi_3(-u'''(0)t^3-2u''(0)t^2-2u'(0)t-2u(0)) & \varphi_3(t) & \varphi_3(1)\\ \varphi_4(t^3) & \varphi_4(-u'''(0)t^3-2u''(0)t^2-2u'(0)t-2u(0)) & \varphi_4(t) & \varphi_4(1)\end{vmatrix}=-3u''(0)\Delta \qquad (3)$$

$$\Delta_3(Lu)=\begin{vmatrix}\varphi_1(t^3) & \varphi_1(t^2) & \varphi_1(-u'''(0)t^3-2u''(0)t^2-2u'(0)t-2u(0)) & \varphi_1(1)\\ \varphi_2(t^3) & \varphi_2(t^2) & \varphi_2(-u'''(0)t^3-2u''(0)t^2-2u'(0)t-2u(0)) & \varphi_2(1)\\ \varphi_3(t^3) & \varphi_3(t^2) & \varphi_3(-u'''(0)t^3-2u''(0)t^2-2u'(0)t-2u(0)) & \varphi_3(1)\\ \varphi_4(t^3) & \varphi_4(t^2) & \varphi_4(-u'''(0)t^3-2u''(0)t^2-2u'(0)t-2u(0)) & \varphi_4(1)\end{vmatrix}=-6u'(0)\Delta \qquad (4)$$

and $\Delta_4(Lu) = -6u(0)\Delta$. Also, $\Delta_{ij}, i, j = 1,2,3,4, \Delta_k(v)_{ij}, i,k = 1,2,3,4, j = \{1,2,3,4\} \setminus \{k\}$, are the cofactors of $\varphi_i(t^{4-j})$ in $\Delta, \Delta_k(v), k = 1,2,3,4$ respectively.

Mawhin's continuation theorem:

Let X, Y be the Banach space, $L: domL \subset X \to Y$ be the Linear mapping, $N: X \to Y$ be the Nonlinear continuous mapping, Let $\dim\ker L = \dim(Y/\operatorname{Im} L) < +\infty$, and $\operatorname{Im} L$ is a Closed set in Y, according to L is a Fredholm operator whose index is zero. If L is a Fredholm operator whose index is zero, then there is a continuous projection operator $P: X \to KerL$ and $Q: Y \to Y_1$, such that $\operatorname{Im} P = Ker L, Ker Q = \operatorname{Im} L, X = KerL \oplus KerP, Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. $L_P := L|_{domL \cap X_1}$ is invertible, so let's call that the inverse K. If $QN(\overline{\Omega})$ is bounded, and $K(I-Q)N: \overline{\Omega} \to X$ is relatively tight in X, according to N is $L-$ tight in $\overline{\Omega}$, where Ω is any bounded open set in X.

Theorem 2.1: (Mawhin coincidence degree theory [10]) Let X, Y be the Banach space, L is a Fredholm operator whose index is zero, $N: \overline{\Omega} \to Y$ is $L-$ tight in $\overline{\Omega}$. If

(1) $Lx \neq \lambda Nx, \forall (x,\lambda) \in (domL \cap \partial\Omega) \times (0,1)$;

(2) $Nx \notin \operatorname{Im} L, \forall x \in KerL \cap \partial\Omega$;

(3) $\deg(JQN, \Omega \cap KerL, 0) \neq 0$, there $J: \operatorname{Im} Q \to KerL$ is a linear isomorphism; equation $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$.

We work in $U = C^3[0,1]$ with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty, \|u'''\|_\infty\}$, where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$. we define $V = L^1[0,1]$ with the norm $\|v\|_1 = \int_0^1 v(t)dt$.

In this paper, we always suppose that the following condition holds:

(C) There exist constants $k_i > 0, i = 1,2,3,4$, such that $|\varphi_i(u)| \le k_i \|u\|, u \in U$ and the function $f(t,x,y,z,w)$ satisfies the Carath´eodory conditions, that is, $f(\cdot,x,y,z,w)$ is measurable for each fixed $(x,y,z,w) \in R^4$, $f(t,\cdot,\cdot,\cdot,\cdot)$ is continuous for a.e. $t \in [0,1]$.

3. The main results

In this case, we assume that there exists $j \in \{1,2,3,4\}$ such that $\Delta_{j4} \neq 0$. In what follows, we choose and fix such j.

Lemma 3.1[12] There exists a function $g_4 \in V$ such that $\Delta_4(g_4) = 1$.

Lemma 3.2[12] $\operatorname{Im} L = \{v \in V : \Delta_4(v) = 0\}$.

Lemma 3.3 $K_{P4} = (L|_{domL \cap KerP_4})^{-1}$.

We introduce the constants $l_3 = k_1|\Delta_{14}| + k_2|\Delta_{24}| + k_3|\Delta_{34}| + k_4|\Delta_{44}|$ and

$$l = \max\{k_1k_2, k_1k_3, k_1k_4, k_2k_3, k_2k_4, k_3k_4\}. \tag{5}$$

The next assumption is fulfilled in the main results by virtue of appropriate assumptions on $f(t,\cdot,\cdot,\cdot)$:

(H_1) For any $r > 0$, there exists a function $h_r \in V$ such that $|f(t,u(t),u'(t),u''(t),u'''(t))| \le h_r(t)$, $u \in U, \|u\| \le r$.

Lemma 3.4 There exists a function $g_4 \in V$ such that $\Delta_4(g_4) = 1$.

If (H_1) holds and $\Omega \subset U$ is bounded, then N is L-compact on $\overline{\Omega}$.

In order to obtain the main results, we impose the following conditions:

(H_2) There exist nonnegative functions $a,b,c,d,e \in V$ such that

$\left| f(t,x,y,z,w) \right| \le a(t)+b(t)|x|+c(t)|y|+d(t)|z|+e(t)|w|, t\in[0,1], a,b,c,d,e\in R$;

(H_3) There exists a constant $M_{04}>0$ such that $\Delta_4(Nu)\ne 0$ if $|u(t)|>M_{04}, t\in[0,1]$;

(H_4) There exists a constant $M_{14}>0$ such that if $|c|>M_{14}$,then one of the following two inequalities holds:

$$c\Delta_4(Nc)>0 \tag{6}$$

$$\text{or } \; c\Delta_4(Nc)<0 \tag{7}$$

(here $Nc=f(t,c,0,0,0), c\in R$)

Lemma 3.5[12] Assume that (H_2) (H_3) hold and let

$$A_{P4}(\|b\|_1+\|c\|_1+\|d\|_1+\|e\|_1)<\frac{1}{2}. \tag{8}$$

where $A_{P4}=1+\frac{8l}{|\Delta_{j4}|}$.Then $\Omega_{14}=\{u\in domL\setminus KerL : Lu=\lambda Nu, \lambda\in(0,1)\}$ is bounded.

Lemma 3.6[12] Assume that (H_4) holds. Then $\Omega_{24}=\{u\in KerL : Nu\in \operatorname{Im} L\}$ is bounded.

Lemma 3.7 Assume that (H_4) holds. Then $\Omega_{44}=\{U:\rho\lambda U+(1-\lambda)\Delta_4(Nu)=0, u\in KerL, \lambda\in[0,1]\}$ is bounded, where $\rho=\begin{cases}1, if\,(6) holds\\ -1, if\,(7) holds\end{cases}$.

Theorem 3.1: Assume that (H_2)- (H_4) and (8) hold. Then problem (1) has at least one solution.

Proof Let $\Omega\supset\overline{\Omega}_{14}\cup\overline{\Omega}_{24}\cup\overline{\Omega}_{34}\cup\overline{\Omega}_{44}$ be bounded. It follows Lemmas3.5 and Lemmas3.6 that

$Lu\ne\lambda Nu, u\in(domL\setminus KerL)\cap\partial\Omega, \lambda\in(0,1)$,and $Nu\notin \operatorname{Im} L, u\in KerL\cap\partial\Omega$.Let

$H(u,\lambda)=\lambda\rho u+(1-\lambda)J_4Q_4Nu$,

where $J_4:\operatorname{Im} Q_4\to KerL$ is an isomorphism defined by $J_4(cg_4)=c, c\in R$. By Lemma3.7,we know

$H(u,\lambda)\ne 0, u\in\partial\Omega\cap KerL, \lambda\in[0,1]$. Since the degree is invariant under a homotopy,

$$\begin{aligned}\deg(J_4Q_4N|_{KerL}, \Omega\cap KerL, 0,0) &= \deg(H(\cdot,0),\Omega\cap KerL,0,0)\\ &=\deg(H(\cdot,1),\Omega\cap KerL,0,0)\\ &=\deg(\rho I,\Omega\cap KerL,0,0)\ne 0.\end{aligned}$$

By Theorem2.1, $Lu=Nu$ has a solution in $domL\cap\overline{\Omega}$.

4. Conclusion

In this paper, the existence of at least one solutions to boundary value problem of resonance fourth-order with functional boundary; By means of Machin's continuation theorem, the existence of solution is verified.

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